PFAFFIANS AND DETERMINANTS FOR SCHUR Q-FUNCTIONS

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Abstract

Schur Q–functions were originally introduced by Schur in relation to projective representations of the symmetric group and they can be defined combinatorially in terms of shifted tableaux. In this paper we describe planar decompositions of shifted tableaux into strips and use the shapes of these strips to generate pfaffians and determinants that are equal to Schur Q–functions. As special cases we obtain the classical pfaffian associated with Schur Q–functions, a pfaffian for skew Q–functions due to Jozefiak and Pragacz, and some determinantal expressions of Okada. We also obtain results for Schur P–functions, results for supersymmetric Schur functions, and generalizations to variable sets subscripted by arbitrarily ordered alphabets.

1 Introduction and Background

It is well–known that the Schur Q–function and skew Schur Q–function can be expressed in terms of pfaffians whose terms are determined by the rows in a shifted diagram. Related determinantal results have also recently appeared in Okada [7]. In this paper we generalize these results, planarly decomposing shifted diagrams into geometrical objects called "strips" and using the shapes of these strips to generate pfaffians and determinants. This approach has also been used for classical Schur functions and determinants and these results appear in Hamel and Goulden [4]. For Schur Q–functions we follow the notation of Sagan [9] and Stembridge [13].

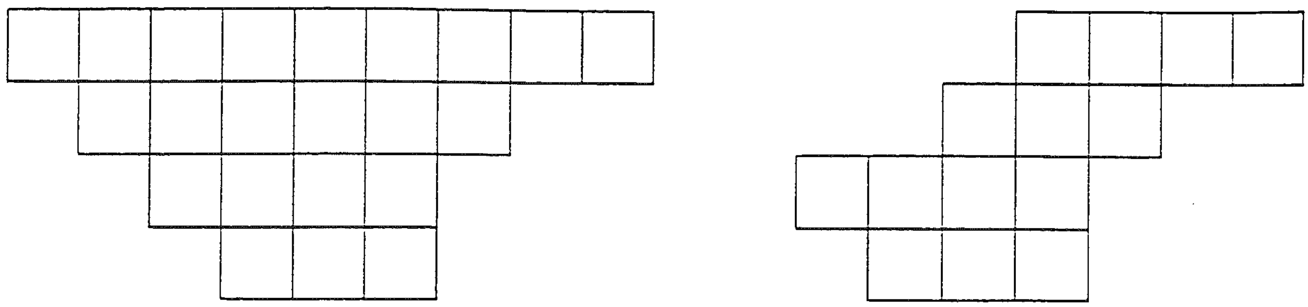


Figure 1: Shifted Tableau and Skew Shifted Tableau

Let λ be a partition of k into l *distinct* positive parts, i.e. $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_l$, where the λ_i are nonnegative integers and $\lambda_1 + \lambda_2 + \ldots + \lambda_l = k$ (λ_i is the ith part of λ) The empty partition $\emptyset$ of 0 has no parts. Let $l(\lambda)$ be the number of parts of λ. Then we can associate with λ a shifted Ferrers diagram (or, simply, shifted diagram), a top justified arrangement of boxes such that row i contains λ_i boxes and has its first box in the ith row and ith column. If we have an additional partition, μ, with distinct parts such that $\mu \subseteq \lambda$, we can define the skew shifted diagram to be the shifted diagram of λ with the shifted diagram of μ removed from the upper left hand corner. i.e. it includes a box in row i, column j iff $\mu_i < j \leq \lambda_i$ and $i \leq j$. The content of a box α in a shifted diagram is the difference $j - i$ where α lies in column j from the left and row i from the top of the diagram (referred to as box (i, j) where convenient).

These shifted diagrams can be filled with integers to create what are known as *tableaux*. Our main interest is Schur Q–functions, thus we will describe the fillings that generate them. Consider an ordered alphabet $1' < 1 < 2' < 2 < 3' < \ldots$ and fill the boxes with elements from this alphabet such that entries weakly increase across the rows and weakly increase down the columns and such that the following two rules are obeyed:

1) For each $k = 1, 2, 3, \ldots$, there is at most one k per column.

2) For each $k = 1, 2, 3, \ldots$, there is at most one k' per row.

Let $T(i, j)$ denote the tableau entry in box (i, j). Define the *profile* of a shifted tableau to be the entries in the main diagonal boxes, i.e. $T(1,1)$, $T(2,2), \ldots, T(n,n)$.

Then the (skew) Schur Q–function, $Q_{\lambda/\mu}(X)$, in the variables $X = (x_1, x_2, \ldots)$ is defined as

$$Q_{\lambda/\mu}(X) = \sum_T \prod_{\alpha \in \lambda/\mu} x_{|T(\alpha)|}, \tag{1}$$

where the summation is over all skew shifted tableaux T of shape λ/μ, where $\alpha \in \lambda/\mu$ means α ranges over all squares in the skew shifted diagram of λ/μ and where $|T(\alpha)| = k$ if $T(\alpha) = k$ or k'. Note $Q_\lambda(X) = Q_{\lambda/\emptyset}(X)$.

The generating function, $q_m(X)$, for a single row in a shifted tableau is defined as

$$\sum_{m \geq 0} q_m(X) t^m = \prod_{i \geq 1} (1 + x_i t) \prod_{j \geq 1} (1 - x_j t)^{-1}.$$

Given an ordered set of objects, $a = (a_1, \ldots a_n)$, we define a 1–factor to be a perfect matching: a set of (undirected) edges on vertex set a such that each a_i is incident with exactly one edge. We use $\mathcal{F}(a)$ to denote the 1–factors of a and $\mathcal{F}_n$ to denote the 1–factors of $\{1, 2, \ldots, n\}$. We will list the edges of a 1–factor in the form (a_i, a_j) where $i < j$ and will define two edges (a_i, a_j) and (a_k, a_l) to be crossed if $i < k < j < l$ or $k < i < l < j$ We define the sign of a 1–factor π, $sgn\ \pi$, to be $(-1)^k$ where k is the number of pairs of crossed edges in π. If $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix, we define the *pfaffian* of A as

$$pf(A) = \sum_{\pi \in \mathcal{F}_n} sgn\ \pi \prod_{(i,j) \in \pi} a_{ij}.$$

Given the pfaffian we can define a well-known identity for Schur Q–functions. This identity may be considered to be a Q–function analogue of the Jacobi–Trudi identity for classical Schur functions. Jozefiak and Pragacz [5] have recently proved a skew version of Theorem 1.1, and Stembridge [13] has proved both of these theorems using lattice paths.

Theorem 1.1 (Stanley [11]) *If λ is a partition consisting of l distinct parts, then*

$$Q_\lambda(X) = \begin{cases} pf[Q_{(\lambda_i, \lambda_j)}]_{1 \leq i < j \leq l} & \text{if } l \text{ is even}, \\ pf[Q_{(\lambda_i, \lambda_j)}]_{1 \leq i < j \leq l+1} & \text{if } l \text{ is odd}, \end{cases}$$

where $\lambda_{l+1} = 0$ if l is odd.

Theorem 1.2 (Jozefiak and Pragacz [5]) *Let λ and μ be partitions with distinct parts and of lengths l and m respectively. If $l + m$ is odd, then define $\lambda_{l+1} = 0$ and replace l by $l + 1$, so that $l + m$ is even. We have*

$$Q_{\lambda/\mu}(X) = pf \begin{bmatrix} Q & H \\ -H^T & 0 \end{bmatrix}$$

where $Q = [Q_{(\lambda_i, \lambda_j)}]$ for $1 \leq i, j \leq l$ and $H = [Q_{\lambda_i - \mu_{m-j+1}}]$ for $1 \leq i \leq l$, $1 \leq j \leq m$.

Theorems 1.1 and 1.2 each give a single pfaffian equal to a given Schur Q–function. Our main result, Theorem 3.1, gives an entire family of pfaffians equal to a given Schur Q–function (the pfaffians of Theorems 1.1 and 1.2 are included in this family). Each of these pfaffians corresponds to a planar geometrical decomposition of the diagram into "strips" to be defined in Section 2. Section 3 contains a lattice path proof of the main result, and the derivations of Theorems 1.1 and 1.2 as corollaries. A determinantal expression for fixed profile also appears in Section 3. In Section 4 we discuss similar results for supersymmetric functions. Section 5 generalizes the results of Section 3 to include different total orderings on the tableau entries.

2 Outside Decompositions of Shifted Tableaux

This section follows the terminology of Hamel and Goulden [4] in which *strip* and *outside decomposition* were defined for standard shape diagrams.

Definition 2.1 *A strip in a shifted diagram of skew shape is a skew diagram with an edgewise connected set of boxes that contains no* 2×2 *block of boxes.*

Strips have a variety of names in the literature, including border strips (Macdonald [6]), skew hooks (Russian edition of [6]) and rim hooks (Sagan [13]).

We employ an "active" vocabulary when referring to strips and boxes. For example a strip "starts" at a box (called the starting box) if that box is the bottommost and leftmost in the strip, and a strip "ends" at a box (called the ending box) if that box is the topmost and rightmost in the strip. A strip "proceeds north" from one box to the one on top of it, and a strip "proceeds east" from one box to the one to the right of it. A box is "approached from the left" if either there is a box immediately to its left or the box is on the left perimeter of the diagram, and a box is "approached from below" if either there is a box immediately below it or the box is on the bottom perimeter of the diagram. See Figure 2 for an example of a strip, where the starting box is marked with a 0 and the ending box is marked with a 1.

Definition 2.2 *Suppose* $\theta_1, \theta_2, \ldots, \theta_m$ *are strips in a skew shifted diagram of* λ/μ *and each strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then if the disjoint union of these strips is the skew diagram of* λ/μ*, we say the totally ordered set* $(\theta_1, \theta_2, \ldots, \theta_m)$ *is a* (planar) outside decomposition *of* λ/μ.

The restrictions of Definition 2.2 force the following property:

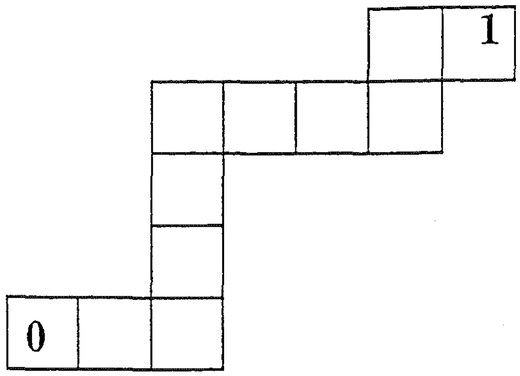


Figure 2: Example of a strip

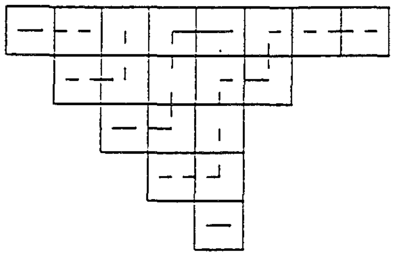


Figure 3: Example of an outside decomposition

Property 2.3 *Boxes of the same content are approached from the same direction in their respective strips; that is, they are either all approached from below or all approached from the left.*

Our main result involves a determinant whose (i, j) entry is defined in terms of strip $\theta_i \# \theta_j$ in a noncommutative way from superimposing θ_i and θ_j. There are two cases in the definition of $\theta_i \# \theta_j$.

Case I: Suppose θ_i and θ_j have some boxes with the same content. Slide θ_i along top–left–to–bottom–right diagonals so that the box of content k in θ_i is superimposed on the box of content k in θ_j for all k. This procedure is well–defined since, as noted above, the two sets of boxes with the same contents are both arranged in the same shape. Define $\theta_i \# \theta_j$ to be the diagram obtained from this superposition by taking all boxes between the ending box of θ_i and the starting box of θ_j inclusive.

Case II: Suppose θ_i and θ_j are two disconnected pieces (and thus do not have any boxes of the same content). The starting box of one will be to the right and/or above the ending box of the other. "Bridge the gap" between θ_i and θ_j by inserting boxes from the ending box of one to the starting box of the other so that these inserted boxes follow the arrangement dictated by other boxes of the same content in the outside decomposition (Property 2.3 ensures the boxes of the same content are arranged in the same way). If there is a content for which there is no box of that content in the diagram (and therefore no determination of the direction from which the box is approached), then bridge those parts of the gap by deciding from which direction a box of this content should be approached and then fixing this choice for that particular diagram. Define $\theta_i \# \theta_j$ as in Case I with the following additional conventions. If the ending box of θ_i is edge connected to the starting box of θ_j, and occurs before it (that is, below or to the left), then $\theta_i \# \theta_j = \emptyset$. If the ending box of θ_i is not edge connected but occurs before the starting box of θ_j, $\theta_i \# \theta_j$ is undefined.

3 Main Result

This section contains the main result of this paper, a result which demonstrates the connection between outside decompositions, pfaffians, and Schur Q–functions. The statement of Theorem 3.1 shows how every outside decomposition of a shifted diagram gives rise to a pfaffian and the proof of Theorem 3.1 further establishes that the pfaffian is equal to the Schur Q–function defined by the shifted diagram that was originally decomposed. The proof is combinatorial and is based on an application of Theorem 3.2 of Stembridge [13]. Before proceeding to the statement of Theorem 3.1, we need to define another type of function below in equation (2). These functions are similar to the Schur Q–functions and are also denoted by a Q but are subscripted by ordered pairs of strips rather than by partitions (this notation is consistent with Stembridge [13]).

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ be an outside decomposition of a shifted diagram. Define ρ to be the strip consisting of the single box in position $(1,1)$. Let I be a set of coordinates of the form $(0, a)$ where $a \in \{0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, \frac{1}{2}', 1', 1\frac{1}{2}', \ldots\}$ and where $(0,a) < (0,b)$ if $a < b$. Define

$$Q_{(\theta_i,\theta_j)}(u_i, u_j) = \sum_{v_1 < v_2 \in I} Q_{\theta_i \# \rho}(u_i, v_1) Q_{\theta_j \# \rho}(u_j, v_2) - Q_{\theta_i \# \rho}(u_i, v_2) Q_{\theta_j \# \rho}(u_j, v_1). \quad (2)$$

where the sum extends to all pairs for which v_1 precedes v_2 in the ordering of I. If $i = j$, note this sum is zero. Also note $Q_{(\theta_j,\theta_i)}(u_j, u_i) = -Q_{(\theta_i,\theta_j)}(u_i, u_j)$ if $i \leq j$.

Theorem 3.1 *Let λ and μ be partitions with distinct parts. Let $\theta = \theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_m$ be an outside decomposition of the shifted diagram λ/μ where θ_i, $1 \leq i \leq k$, includes a box of the main diagonal of the diagram and θ_i, $k+1 \leq i \leq m$, does not. If $2m - k$ is odd, define $\theta_0 = \emptyset$ and replace θ by $\theta_0 \cup \theta$. Then*

$$Q_{\lambda/\mu}(X) = pf \begin{bmatrix} Q & H \\ -H^T & 0 \end{bmatrix},$$

where $Q = [Q_{(\theta_i,\theta_j)}(u_i,u_j)],\ 1 \le i,j, \le m$ *for* $u_i = (d-c+1,0)$ *if strip* i *has ending box on the top perimeter in box* (c,d) *of the shifted diagram or* $u_i = (d-c+1,\infty)$ *if strip* i *has ending box on the right perimeter in box* (c,d) *of the diagram* ($u_i = (d-c+1,0)$ *if both),* $i = 1,\ldots,m$, *and* $H = [\dot{Q}_{\theta_i \# \theta_j}],\ 1 \le i \le m,\ k+1 \le j \le m$.

We delay the proof to describe the lattice path environment. Label the y–axis with $1', 1, 2', 2, \ldots$. Define lattice paths with five types of permissible steps: up–vertical steps that increase the y–coordinate by 1; down–vertical steps that decrease the y–coordinate by 1; right–horizontal steps (called horizontal) that increase the x–coordinate by 1; up–diagonal steps from unprimed levels to primed levels that increase the x–coordinate by 1 and increase the y–coordinate by 1; and down–diagonal steps from primed levels to unprimed levels that increase the x–coordinate by 1 and decrease the y–coordinate by 1. We also distinguish between horizontal steps at primed levels and horizontal steps at unprimed levels. The steps are subject to the following additional restrictions: a down–vertical step must not precede an up–vertical step, an up–vertical step must not precede a down–vertical step, an up–vertical step must not precede a horizontal step at a primed level, an up–vertical step must not precede a down–diagonal step, a down–vertical step must not precede a horizontal step at an unprimed level, and a down–vertical step must not precede an up–diagonal step. We also require that all steps between lines $x = c$ and $x = c+1$ for all c are either 1) horizontal at primed levels or down–diagonal, or 2) horizontal at unprimed levels or up–diagonal. The determination of whether the steps are of type 1) or 2) is made by the outside decomposition: if boxes of content d are approached from the left, then steps between $x = d$ and $x = d+1$ must be of type 2); if the boxes of content d are approached from below, then steps between $x = d$ and $x = d+1$ must be of type 1).

Proof of Theorem 3.1: Fix starting points $I = \{(0,\frac{1}{2}), (0,1'), (0,1\frac{1}{2}'), (0,1), (0,1\frac{1}{2}), (0,2'), (0,2\frac{1}{2}'), \ldots\}$ and $v_{k+i} = (t-s,0)$ if strip $k+i$ has starting box on the left perimeter in box (s,t) of the shifted diagram, or $k+i = (t-s,\infty)$ if strip $k+i$ has starting box on the bottom perimeter in box (s,t) of the shifted diagram ($v_{k+i} = (t-s,0)$ if both), $i = 1,\ldots,m-k$. Fix ending points $u_i = (d-c+1,0)$ if strip i has ending box on the top perimeter in box (c,d) of the shifted diagram or $u_i = (d-c+1,\infty)$ if strip i has ending box on the right perimeter in box (c,d) of the diagram ($u_i = (d-c+1,0)$ if both), $i = 1,\ldots,m$. If $2m+k$ is odd, adjoin the "phantom vertex" $u_{m+1} = (0,\infty)$ for θ_0 and let $I = I \cup \{u_{m+1}\}$.

Given a shifted tableau of shape λ/μ with an outside decomposition, we can construct an m–tuple of nonintersecting lattice paths. For each strip construct a path as follows: if a box containing i and at coordinates (a,b) in the shifted diagram is approached from the left in the strip, put a horizontal step from $(a-b,i)$ to $(a-b+1,i)$; if it is approached from below, put a down–diagonal step from $(a-b,(i+1)')$ to

$(a-b+1,i)$. If a box containing i' and at coordinates (a,b) in the shifted diagram is approached from the left in the strip, put an up–diagonal step from $(a-b,i-1)$ to $(a-b+1,i')$; if it is approached from below, put a horizontal step from $(a-b,i')$ to $(a-b+1,i')$. Connect these steps with vertical steps. It is routine to verify that there is a unique way of doing this.

Now we verify that if an m–tuple of lattice paths is intersecting, it does not correspond to a tableau. The essential reasons for this are the row and column weakness conditions in the tableau and the restriction that only one k' occurs in any row and only one k occurs in any column. We now give a detailed consideration.

Suppose on the contrary that there is some intersecting m–tuple of lattice paths that corresponds to a tableau. Then we will show that we obtain a contradiction by considering the first intersection point from the left. The requirement that steps between $x=c$ and $x=c+1$ are of the same type means that the following types of intersection are not possible: a down–diagonal step from (a,b') to $(a+1,b-1)$ and a horizontal step from $(a,b-1)$ to $(a+1,b-1)$, a down–diagonal step from (a,b') to $(a+1,b-1)$ and a horizontal step from (a,b') to $(a+1,b')$, an up–diagonal step from (a,b) to $(a+1,(b+1)')$ and a horizontal step from (a,b) to $(a+1,b)$ and, finally, an up–diagonal step from (a,b) to $(a+1,(b+1)')$, and a horizontal step from $(a,(b+1)')$ to $(a+1,(b+1)')$. The row and column weakness requirements and the restrictions on consecutive k's and (k')'s imply the tableau is also diagonal strict; that is, the entries increase along top–left–to–bottom–right diagonals, or, equivalently, entries with the same content are strictly increasing. This demonstrates it is not possible for two boxes of the same content to generate two steps at the same position in the plane.

Any intersection must involve either an up–vertical step and a down–vertical step, a horizontal step (either at a primed or unprimed level) and an up–vertical step, a horizontal step and a down–vertical step, a down–diagonal step and an up–vertical step, a down–diagonal step and a down–vertical step, a down–diagonal step and a down–vertical step, an up–diagonal step and an up–vertical step, or an up–diagonal step and a down–vertical step. Intersections between up– or down–diagonal steps and horizontal steps (at primed or unprimed levels) will be subsumed by these cases since the steps in the second path must be preceded by an up–vertical or down–vertical step (i.e. the restriction that steps between $x=c$ and $x=c+1$ are of same type and the restriction that boxes of the same content cannot generate two steps at the same position in the plane guarantee this). Consider a number of cases.

Case I (An up–vertical step in path i intersects a down–vertical step in path j and neither path has nonvertical steps before the x–coordinate of the intersection point): Both of the paths must have nonvertical steps. Then Case I cannot possibly occur since the first nonvertical step in the path that starts at y–coordinate ∞ must be diagonal, and the first nonvertical step in the path that starts at y–coordinate 1 must be horizontal, but these first nonvertical steps must both occur between $x = c$ and $x = c + 1$ for some c.

Case II (**A horizontal step at height a in path i intersects an up–vertical step in path j; path j has a step ending at height d (or d') before the up–vertical steps and a step ending at height e (or e') after the up–vertical steps**): The content of the box containing e (or e') is one more than the content of the box containing a, and $e \geq a$, so the box containing e (or e') is right and below (or beside) of the box containing a by column and row weakness and restrictions on the number of occurrences of integers in rows and columns of the tableau. The content of the box containing d (or d') is the same as the content of the box containing a, and $d < a$, so the box containing d (or d') is above and to the left of the box containing a by column and row weakness and the restrictions on the number of occurrences of integers in rows and columns of the tableau. But the box containing d (or d') and the box containing e (or e') are in the same strip, yet located on different sides of the box containing a. This provides a contradiction.

Case III (**A horizontal step at height a in path i intersects an up–vertical step in path j; path j has a step at height d (or d') before the up–vertical steps and no nonvertical steps after**): Since there are no nonvertical steps after, path j ends at y–coordinate ∞ and the corresponding strip ends on the right perimeter. But as in Case II, the box containing d (or d') is to the left and above the box containing a, so it is not possible for the strip to end on the right perimeter, and we obtain a contradiction.

Case IV (**A horizontal step at height a in path i intersects an up–vertical step in path j; path j has a step at height e (or e') after the up–vertical steps and no nonvertical steps before**): Since there are no nonvertical steps before, path j starts at y–coordinate 0 and the corresponding strip starts on the left perimeter. But as in Case II, the box containing e (or e') is below (or beside) and to the right of the box containing a, so it is not possible for the strip to start on the left perimeter, and we obtain a contradiction.

There are additional cases similar to Cases II to IV with "up–vertical" replaced by "down–vertical," and others with "horizontal" replaced by "up–diagonal," "down–diagonal," or "horizontal at height a'." The arguments for these remaining cases are similar to the arguments for Cases II to IV. Hence tableaux correspond only to nonintersecting m–tuples of lattice paths.

The construction described above for generating paths given tableaux is reversible, and now we verify that a nonintersecting m–tuple of lattice paths obeying these conditions corresponds to a shifted tableau with the given outside decomposition. The choice of the starting and ending points and the restrictions on the steps ensures that the m–tuple corresponds to the shifted diagram of the partition, but we must show entries obey the rules governing tableaux for Schur Q–functions.

We begin by ensuring that a lattice path that starts at v_j or at an element in I and ends at u_i corresponds to the strip $\theta_i \# \theta_j$. The proof is as follows. Begin with the empty partition. At iteration k, if the kth nonvertical step from the left is horizontal ending at (i, j), then place a box containing j in the tableau to the right of the previous box. If it is horizontal ending at (i, j'), then place a box containing j' on top of the previous box. If it is down–diagonal ending at (i, j), then place a box containing j in the tableau on top of the previous box. If it is up–diagonal ending at (i, j'), then place a box containing j' in the tableau beside the previous box. The fact that a down–vertical step precedes neither a horizontal step at an unprimed level nor an up–diagonal step ensures that these steps end at a height higher than or the same as the step just before it. This means the entries in a row of the tableau are weakly increasing. The fact that an up–vertical step precedes neither a down–diagonal step nor a horizontal step at a primed level ensures that these steps end at a height higher than or the same as the step just before it. This means entries in a column of the tableau are weakly increasing. Since the tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from θ_j and θ_i, since boxes of the same content correspond to the same type of step, and since the $\#$ operation is based on boxes of the content, we know the strip is $\theta_i \# \theta_j$.

Let $T(l, j)$ denote the entry in box (l, j) of the tableau. It is routine to verify that the presence of up–diagonals limits us to at most one j' per row, while the presence of down–diagonals limits us to at most one j per column. We claim $T(l, j) \leq T(l, j+1)$ (row weakness). Suppose the step α between $x = c$ and $x = c + 1$ corresponding to $T(l, j)$ ends at height t (resp. t'). If step β between $x = c + 1$ and $x = c + 2$ corresponding to $T(l, j + 1)$ is in the same path, then β must be a horizontal step if $T(l, j + 1)$ is unprimed, and must be an up–diagonal step if $T(l, j + 1)$ is primed. Since these two types of steps are preceded by up–vertical steps if by any verticals at all, they must occur at a height greater than or equal to t (resp. t').

Suppose now β is in a different path. Since α ends at t (resp. t'), β must start at a height greater than or equal to height $(t+1)'$ (resp. t) to avoid intersection. So $T(l, j) = t$ (resp. t'), $T(l, j + 1) \geq t$ (resp. t), i.e. it must start at $(t + 1)'$ (resp. t) but could be down–diagonal and end at t (resp. horizontal and end at t). So $T(l, j) \leq T(l, j + 1)$.

We claim $T(l, j) \leq T(l + 1, j)$ (column weakness). Suppose the step α between $x = c$ and $x = c + 1$ corresponding to $T(l, j)$ ends at height t (resp. t'). If the step β between $x = c - 1$ and $x = c$ corresponding to $T(l + 1, j)$ is in the same path, then β must end above or on the same level as α, since β is approached from above or at the same height in the path. Hence $T(l + 1, j) \geq T(l, j)$.

Suppose now β is in a different path. Since α starts at $(t+1)'$, t, or t' (resp. t, t', or $(t-1)$), β must end at a height greater than or equal to t (resp. t') to avoid intersection. So $T(l,j) = t$ (resp. t'), $T(l+1,j) \geq t$ (resp. t'). So $T(l,j) \leq T(l+1,j)$.

Note that if we did not allow the half steps on the y–axis, then a pair of paths whose first steps were, for example, $(0,1) \to (1,2')$ and $(0,1) \to (1,1)$ would necessarily intersect. However this corresponds to a legitimate tableau configuration of the form

$$\begin{matrix} 1 & * \\ & 2' \end{matrix}$$

Note that this arrangement would not be legitimate within the tableau, for we could not place an integer below 1 and before $2'$.

For each horizontal or diagonal step ending at (i,j) or (i,j'), choose a weight of x_j. For each vertical step, regardless of position, choose a weight of one. Since there is a one–to–one correspondence between lattice paths and tableaux whose shape is a strip, the generating function for these lattice paths is the Schur Q–function for the shape of the strip.

The proof now follows by the well–known Gessel–Viennot lattice path procedure as described in Gessel–Viennot [2], Goulden and Jackson [3, sec. 5.4], or Sagan [13]. This procedure was originally created to prove the Jacobi–Trudi identity for Schur functions, and it defines a sign–reversing, weight–preserving involution on intersecting m–tuples of lattice paths, thus demonstrating that their contribution to the determinantal sum is zero. To obtain the full generality we require, we invoke the broader result of Stembridge [13, Theorem 3.2]. To do so we must verify that any m-tuple such that u_{k+i} is *not* matched to v_{k+i} for $i = 1, \ldots, m-k$ or such that u_i is matched to $(0,t)$ (where t could stand for a primed integer) and u_j is matched to $(0,s)$ (where s could stand for a primed integer) for $1 \leq i < j \leq k$, $s < t$, necessarily contains an intersection; however, this is routine. Note additionally that although Stembridge does not impose conditions on which steps may follow each other (as we do before this proof), his theorem is still applicable since it is stated in terms of generating functions and is without reference to specific types of steps allowed. ◇

The following matrices are of the type produced by Theorem 3.1. The corresponding outside decomposition and lattice paths are given below in Figure 5. Note the shape decomposed is $\lambda = (9,6,4,2)$, with $\theta_1 = 1$, $\theta_2 = 1,1$, $\theta_3 = 3,1,1$, $\theta_4 = 3,3,1,1/2$, $\theta_5 = 2,1$, $\theta_6 = 3,1$ and $u_1 = (1,0)$, $u_2 = (2,0)$, $u_3 = (5,0)$, $u_4 = (6,0)$, $u_5 = (4,\infty)$, and $u_6 = (9,\infty)$. In order to shorten the width of the matrix we

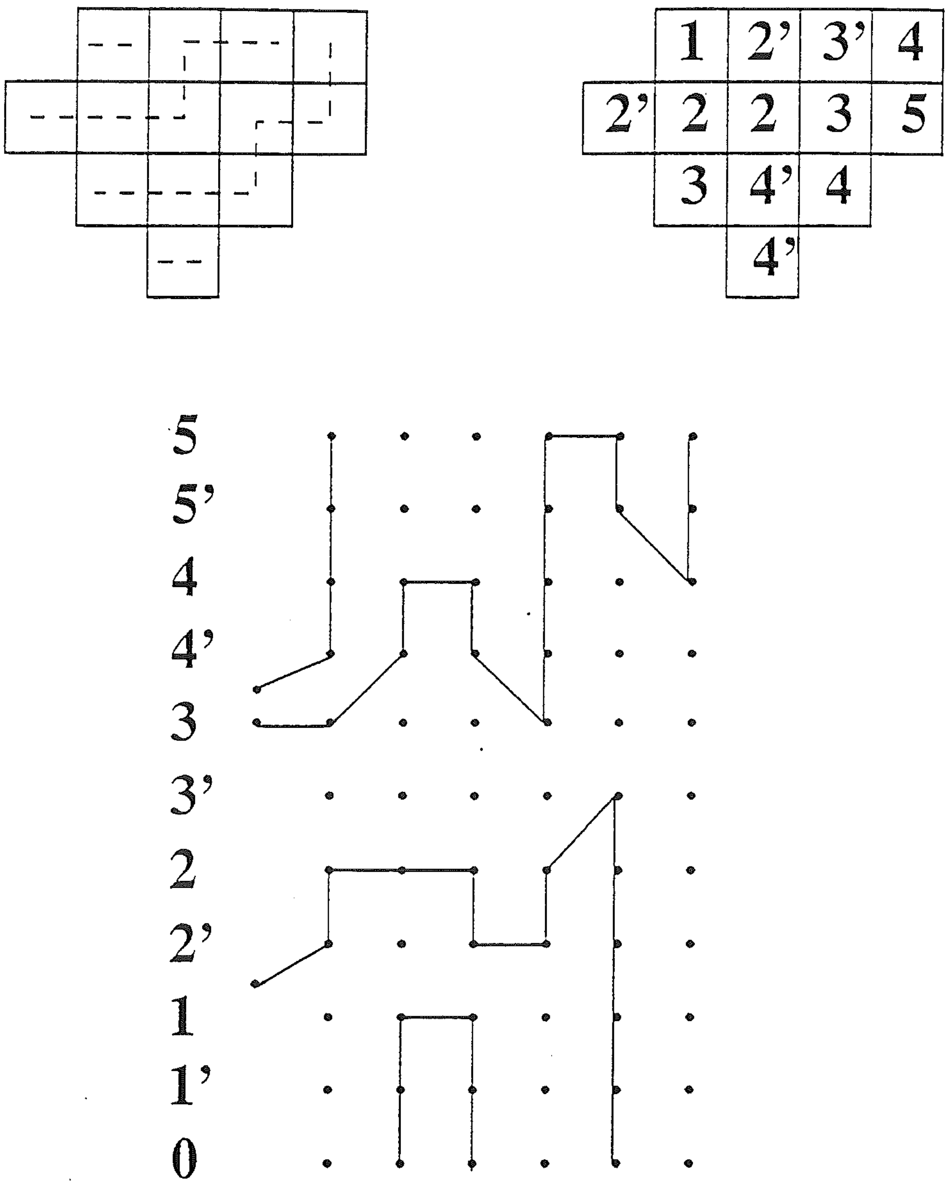


Figure 4: A 4–tuple of lattice paths that illustrates the theorem.

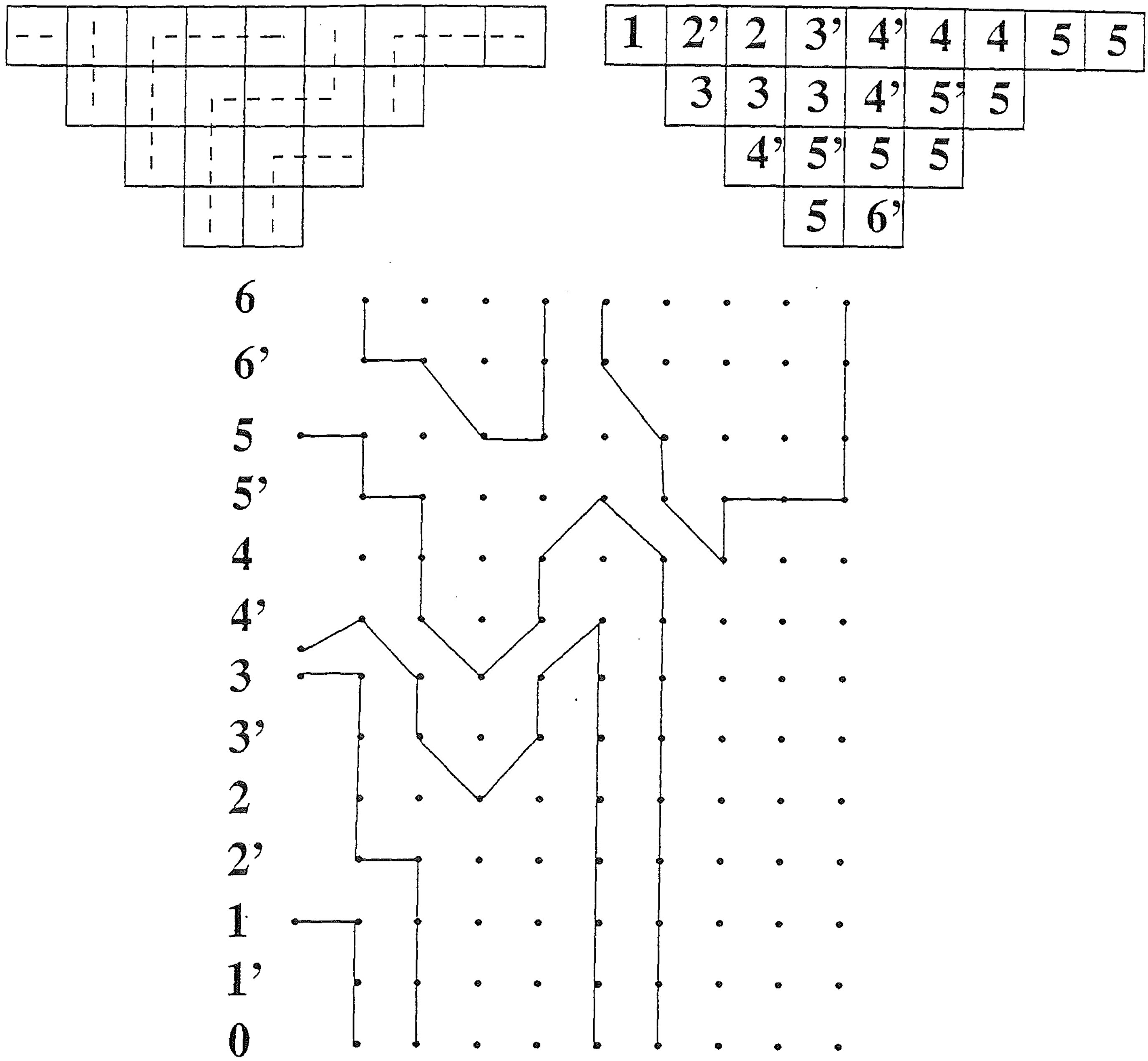


Figure 5: A 6–tuple of lattice paths that illustrates the theorem.

have omitted the (u_i, u_j) portions.

$$Q = \begin{pmatrix} 0 & Q_{1,11} & Q_{1,311} & Q_{1,3311/2} & Q_{1,21} & Q_{1,31} \\ -Q_{1,11} & 0 & Q_{11,311} & Q_{11,3311/2} & Q_{11,21} & Q_{11,31} \\ -Q_{1,311} & -Q_{11,311} & 0 & Q_{311,3311/2} & Q_{311,21} & Q_{311,31} \\ -Q_{1,3311/2} & -Q_{11,3311/2} & -Q_{311,3311/2} & 0 & Q_{3311/2,21} & Q_{3311/2,31} \\ -Q_{1,21} & -Q_{11,21} & -Q_{311,21} & -Q_{3311/2,21} & 0 & Q_{21,31} \\ -Q_{1,31} & -Q_{11,31} & -Q_{311,31} & -Q_{3311/2,31} & -Q_{21,31} & 0 \end{pmatrix}.$$

$$H = \begin{pmatrix} 1 & 0 \\ Q_1 & 0 \\ Q_{31} & 1 \\ Q_{331/2} & Q_1 \\ Q_{21} & 0 \\ Q_{5321/22} & Q_{31} \end{pmatrix}.$$

As stated in the introduction, Theorem 3.1 has two well–known corollaries. They appeared in Section 1 as Theorems 1.1 and 1.2.

Corollary 3.2 (Stanley [11]) *If λ is a partition consisting of l distinct parts, then*

$$Q_\lambda = \begin{cases} pf[Q_{(\lambda_i,\lambda_j)}]_{1\le i<j\le l} & \text{if } l \text{ is even,} \\ pf[Q_{(\lambda_i,\lambda_j)}]_{1\le i<j\le l+1} & \text{if } l \text{ is odd,} \end{cases}$$

where $\lambda_{l+1} = 0$ if l is odd.

Proof: Theorem 3.1 with outside decomposition $\theta = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = \emptyset$. $\diamond$

Corollary 3.3 (Jozefiak and Pragacz [5]) *Let λ and μ be partitions with distinct parts and of lengths l and m respectively. If $l + m$ is odd, then define $\lambda_{l+1} = 0$ and replace l by $l + 1$, so that $l + m$ is even. We have*

$$Q_{\lambda/\mu} = pf \begin{bmatrix} Q & H \\ -H^T & 0 \end{bmatrix}$$

where $Q = [Q_{(\lambda_i,\lambda_j)}]$ for $1 \le i, j \le l$ and $H = [Q_{\lambda_i - \mu_{m-j+1}}]$ for $1 \le i \le l$, $1 \le j \le m$.

Proof: Theorem 3.1 with outside decomposition $\theta = (\lambda_1 - \mu_1, \ldots, \lambda_m - \mu_m, \lambda_{m+1}, \ldots, \lambda_l)$. $\diamond$

Using the same lattice path set–up as in the proof of Theorem 3.1 we can derive a determinantal result for Schur Q–functions. Some determinantal results for Schur Q–functions—and indeed for more general types of functions—appear in Okada [7]. They are stated in terms of plane partitions and concern only decompositions into rows. The results presented here (and those presented in Section 5) are generalizations of this work of Okada [7] to arbitrary skew shape and to arbitrary outside decompositions. Note that the result given below is for fixed profile.

Theorem 3.4 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with distinct nonzero parts, and let $a = (a_1, a_2, \ldots, a_{l-m})$ be a strictly increasing sequence of elements chosen from $1' < 1 < 2' < 2 < \ldots$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be an outside decomposition of λ/μ where $\theta_1 \ldots \theta_{l-m}$ contain some a_i, and the remaining θ_i do not. Then*

$$Q_{\lambda/\mu}(X) =$$
$$\det\left[\left(x_{|a_j|}Q_{\theta_i\#\theta_j}(x_{|a_j|}, x_{(|a_j|+1)'}, x_{|a_j|+1}, \ldots)\right)_{1\le i\le k; 1\le j\le l-m} \vdots \left(Q_{\theta_i\#\theta_j}(X)\right)_{1\le i\le k; l-m+1\le j\le k}\right].$$

Proof: Use the same lattice path set-up as for Theorem 3.1 except that paths that correspond to strips containing a box from the main diagonal of the diagram are constrained to have as first step a horizontal step from $(0, \alpha)$ to $(1, \alpha)$ where α is the element in the box on the main diagonal. A Gessel–Viennot argument similar to that of Theorem 3.1 but designed for determinants (eg. the sign comes from a permutation, not a 1–factor) provides a proof. This can be accomplished by invoking Theorem 1.2 of Stembridge [13]. $\diamond$

If we sum over all permissible sequences a we obtain a more general result.

Corollary 3.5 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with distinct nonzero parts, and let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be an outside decomposition of λ/μ. Then,*

$$Q_{\lambda/\mu}(X) = \sum_{a_1 < a_2 < \ldots < a_{l-m}} \det\left[A \vdots B\right],$$

where

$$A = \left(x_{|a_j|}Q_{\theta_i\#\theta_j}(x_{|a_j|}, x_{(|a_j|+1)'}, x_{|a_j|+1}, \ldots)\right)_{1\le i\le k; 1\le j\le l-m}$$

and

$$B = \left(Q_{\theta_i\#\theta_j}(X)\right)_{1\le i\le k; l-m+1\le j\le k},$$

and where the sum is over all strictly increasing sequences $a = (a_1, \ldots a_{l-m})$ of integers chosen from $1' < 1 < 2' < 2 < \ldots$.

As mentioned above, Okada [7] contains a special case of Corollary 3.5. This result is stated in the language of plane partitions but may be restated in terms of standard shape partitions as Corollary 3.6 below.

Corollary 3.6 (Okada [7]) *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a standard shape partition. Then*

$$Q_\lambda(X) = \sum_{a_1 < a_2 < \ldots < a_l} \det\left[x_{|a_j|} Q_{\lambda_i}(x_{|a_j|}, x_{(|a_j|+1)'}, x_{|a_j|+1}, \ldots)\right].$$

Proof: Corollary 3.5 with outside decomposition $\theta = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = \emptyset$. $\diamond$

Schur P–functions are closely related to Schur Q–functions. Define the Schur P–function, $P_{\lambda/\mu}(X)$, as

$$P_{\lambda/\mu}(X) = \sum_{T^*} \prod_{\alpha \in \lambda/\mu} x_{|T^*(\alpha)|}, \tag{3}$$

where T^* is the set of all skew shifted tableaux with unprimed integers on the main diagonal (i.e. unprimed profile). Then it is easy to see that $P_{\lambda/\mu}(X) = 2^{-l(\lambda)+l(\mu)} Q_{\lambda/\mu}(X)$, since for each a on the main diagonal of the tableau, the choice of either a or a' yields a tableau that generates $Q_{\lambda/\mu}$.

The result of Theorem 3.1 is also valid for Schur P–functions. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ be an outside decomposition of a shifted diagram. Recall ρ is the strip consisting of the single box in position $(1,1)$. Let I be the same set as in Theorem 3.1. Define below in equation (4) a function related to the Schur P–function defined in (3) but defined for an ordered pair of strips rather than for a partition.

$$P_{(\theta_i,\theta_j)}(u_i, u_j) = \sum_{v_1 < v_2 \in I} P_{\theta_i \# \rho}(u_i, v_1) P_{\theta_j \# \rho}(u_j, v_2) - P_{\theta_i \# \rho}(u_i, v_2) P_{\theta_j \# \rho}(u_j, v_1). \tag{4}$$

where the sum extends to all pairs for which v_1 precedes v_2 in the ordering of I. If $i = j$ note this sum is zero. Also note $P_{(\theta_j,\theta_i)}(u_j, u_i) = -P_{(\theta_i,\theta_j)}(u_i, u_j)$ if $i \leq j$.

Theorem 3.7 *Let λ and μ be partitions with distinct parts. Let $\theta = \theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_m$ be an outside decomposition of the shifted diagram λ/μ where θ_i, $1 \leq i \leq k$, includes a box of the main diagonal of the diagram and θ_i, $k+1 \leq i \leq m$, does not. If $2m - k$ is odd, define $\theta_0 = \emptyset$ and replace θ by $\theta_0 \cup \theta$. Then*

$$P_{\lambda/\mu}(X) = pf\begin{bmatrix} P & J \\ -J^T & 0 \end{bmatrix}$$

where $P = [P_{(\theta_i,\theta_j)}(u_i, u_j)]$, $1 \le i, j, \le m$ *for* $u_i = (d-c+1, 0)$ *if strip* i *has ending box on the top perimeter in box* (c, d) *of the shifted diagram or* $u_i = (d-c+1, \infty)$ *if strip* i *has ending box on the right perimeter in box* (c, d) *of the diagram (*$u_i = (d-c+1, 0)$ *if both),* $i = 1, \ldots, m$, *and*

$$J = \begin{bmatrix} (P_{\theta_i \# \theta_j})_{1 \le i \le k;\, k+1 \le j \le m} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (Q_{\theta_i \# \theta_j})_{k+1 \le i \le m; k+1 \le j \le m} \end{bmatrix}.$$

Proof: Use the same lattice path set-up as for Theorem 3.1 except that paths that correspond to strips containing a box from the main diagonal of the diagram are constrained to have their first step ending at an unprimed level. The Gessel–Viennot argument of Theorem 3.1 provides the proof. $\diamond$

We can also specify determinantal results for P–functions. Note that in this case the profile is constrained to contain only unprimed integers.

Theorem 3.8 *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *and* $\mu = (\mu_1, \ldots, \mu_m)$ *be partitions with distinct nonzero parts, and let* $a = (a_1, a_2, \ldots, a_{l-m})$ *be a strictly increasing sequence of positive integers. Let* $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ *be an outside decomposition of* λ/μ *where* $\theta_1 \ldots \theta_{l-m}$ *contain some* a_i, *and the remaining* θ_i *do not. Then*

$$P_{\lambda/\mu}(X) = \det \begin{bmatrix} & \vdots & B \\ A & \vdots & \ldots \\ & \vdots & C \end{bmatrix}$$

where

$$A = \left[\left(x_{a_j} P_{\theta_i \# \theta_j}(x_{a_j}, x_{(a_j+1)'}, x_{a_j+1}, \ldots) \right)_{1 \le i \le k; 1 \le j \le l-m} \right],$$

$$B = \left[\left(P_{\theta_i \# \theta_j}(X) \right)_{1 \le i \le l-m; l-m+1 \le j \le k} \right],$$

$$C = \left[\left(Q_{\theta_i \# \theta_j}(X) \right)_{l-m+1 \le i \le k; l-m+1 \le j \le k} \right].$$

Proof: Use the same lattice path set-up as for Theorem 3.4 except that steps from $x = 0$ to $x = 1$ must end at an unprimed level. The Gessel–Viennot argument of Theorem 3.1 provides a proof. $\diamond$

Corollary 3.9 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with distinct nonzero parts, and let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be an outside decomposition of λ/μ. Then,*

$$P_{\lambda/\mu}(X) = \sum_{a_1<a_2<\ldots<a_{l-m}} = \det \begin{bmatrix} & \vdots & B \\ A & \vdots & \ldots \\ & \vdots & C \end{bmatrix}$$

where A, B, and C are as defined in Theorem 3.8, and where the sum is over all strictly increasing sequences $a = (a_1, \ldots a_{l-m})$ of positive integers.

4 Supersymmetric Functions

In this section we prove a result similar to Theorem 3.1 but for supersymmetric functions. Such a result can also be derived directly from Theorem 3.1 of Hamel and Goulden [4] which is a decomposition result for classical Schur functions and skew shape (not shifted) diagrams. By partitioning the variable set into x's and y's and applying the operator ω_y (where ω_y is defined as $\omega_y h_k(y) = e_k(y)$), the decomposition result of Hamel and Goulden [4] for ordinary symmetric functions can be transformed to a result for supersymmetric functions. However, the same supersymmetric result can also be proved using the constructs established above, thus illustrating the relationship between Schur Q–functions and supersymmetric Schur functions.

Define a *standard shape diagram* of shape λ to be a top and left justified set of boxes with λ_i boxes in the ith row. Define a *skew shape diagram* of shape λ/μ to be the diagram of shape λ with the diagram of μ removed from the upper left hand corner. A standard shape (resp. skew shape) tableau is a standard (resp. skew shape) diagram filled with $1' < 1 < 2' < 2 < \ldots$ such that the entries increase weakly in rows and columns and such that

1) For each $k = 1, 2, 3, \ldots$, there is at most one k per column.

2) For each $k = 1, 2, 3, \ldots$, there is at most one k' per row.

Define the complete supersymmetric function, $h_k(X/Y)$, $k \geq 0$, as

$$\sum_{k\geq 0} h_k(X/Y)t^k = \prod_{i\geq 1}(1 - x_i t)^{-1} \prod_{j\geq 1}(1 + y_j t), \quad k \geq 1,$$

and $h_0(X/Y) = 0$.

Define the supersymmetric Schur function, $s_{\lambda/\mu}(X/Y)$, as

$$s_{\lambda/\mu}(X/Y) = \sum_T \prod_{\alpha \in T} x_\alpha^{m(\alpha)} \prod_{\beta' \in T} y_{\beta'}^{m(\beta')}.$$

where the sum is over all tableau T of shape λ/μ, the first product is over all unprimed integers α in T, the second product is over all primed integers β' in T, and $m(\alpha)$ (resp. $m(\beta')$) is the multiplicity of α (resp. β'), i.e. the number of times α (resp. β') appears in a box of the tableau.

The usual definition of the supersymmetric Schur function (as found, for example, in Berele and Regev [1]) uses the total order $1 < 2 < 3 < \ldots < 1' < 2' < 3' < \ldots$. However, either total order produces the same supersymmetric function. In fact, *any* total order of $\{1, 2, 3, \ldots, 1', 2', 3' \ldots\}$—with the proviso that the natural total orders $1 < 2 < 3 < \ldots$ and $1' < 2' < 3' \ldots$ are preserved (i.e. a total order in which $\{1, 2, 3, \ldots\}$ and $\{1', 2', 3', \ldots\}$ are chains)—will produce the same supersymmetric function. This is a result of Stanley [12] for the standard shape case. The skew shape case is an easy extension of Stanley's argument, and it has appeared in Worley [14, p. 30]. We outline the skew shape case here.

The key point to the proof is that the coefficient of $x_1^{a_1} \ldots x_m^{a_m} y_1^{b_1} \ldots y_n^{b_n}$ in $s_{\lambda/\mu}(X/Y)$ is equal to the coefficient of $s_{\lambda/\mu}(X)$ when the product $h_{a_1} \ldots h_{a_m} e_{b_1} \ldots e_{b_n}$ is expanded as a linear combination of Schur functions. Since the order of the factors in $h_{a_1} \ldots h_{a_m} e_{b_1} \ldots e_{b_n}$ is immaterial, the order of $\{1, 2, \ldots, 1', 2', \ldots\}$ is also immaterial. We use an inner product argument and must show $<< h_a(x) h_b(y), s_{\lambda/\mu}(X/Y) >_x>_y$ equals $< s_{\lambda/\mu}, h_{a_1} \ldots h_{a_m} e_{b_1} \ldots e_{b_n} >_x$. Expand $s_{\lambda/\mu}(X/Y)$ as $< s_\mu(Z), \omega_y s_\lambda(X, Y, Z) >_z$, and complete the proof by inner product manipulations on $< s_\mu(Z), < h_a(X) < e_b(y), s_\lambda(X, Y, Z) >_y>_x>_z$, using (5.9) and (5.1) of Macdonald [6].

Strips and outside decompositions for standard and skew shape diagrams have been given in Hamel and Goulden [4] and are defined in a manner analogous to that for shifted and skew shifted diagrams. This result can also be adapted to account for the null strips described in Hamel and Goulden [4].

The following is the main result of this section. An example appears in Figure 6.

Theorem 4.1 *Let $\theta_1, \theta_2, \ldots, \theta_m$ be an outside decomposition of the skew shape partition λ/μ. Then*

$$s_{\lambda/\mu}(X/Y) = \det(s_{\theta_i \# \theta_j}(X/Y)).$$

Proof: Use the same lattice path set–up as for Theorem 3.1 except that paths have definite starting and ending points prescribed by the contents of the starting boxes and ending boxes of the strips, i.e. fix $v_i = (t-s, 1)$ if strip i has starting box on the left perimeter in box (s,t) of the diagram, or $v_i = (t-s, \infty)$ if strip i has starting box on the bottom perimeter in box (s,t) of the diagram ($v_i = (t-s, 1)$ if both), $i = 1, \ldots, m$; fix points $u_i = (d-c+1, 1)$ if strip i has ending box on the top perimeter in box (c,d) of the diagram, or $u_i = (d-c+1, \infty)$ if strip i has ending box on the right perimeter in box (c,d) of the diagram ($u_i = (d-c+1, \infty)$ if both), $i = 1, \ldots, m$. Then the proof follows by a Gessel–Viennot lattice path argument for determinants similar to that invoked for Theorem 3.4. $\diamond$

In the classical case of the Schur functions, there is a well-known identity that relates a Schur function to a determinant of complete symmetric functions. This identity is called the *Jacobi–Trudi identity* and comes from decomposing a diagram into strips which are the rows of the diagram. There is a supersymmetric version of the Jacobi–Trudi identity as well, also generated by the rows in a diagram, and we state it below.

Corollary 4.2 (Supersymmetric Jacobi–Trudi (Stanley [12]; Remmel [8])) *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions. Then*

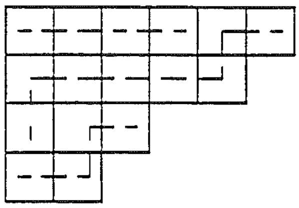
$$s_{\lambda/\mu}(X/Y) = \det(h_{\lambda_i - \mu_j - i + j}(X/Y))_{l \times l}.$$

Proof: Theorem 4.1 with outside decomposition $\theta = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_m - \mu_m, \lambda_{m+1}, \ldots, \lambda_l)$. $\diamond$.

5 Other Total Orders

The total order on $\{1, 2, \ldots, 1', 2', \ldots\}$ chosen in Section 3 is not the only total order for which a version of Theorem 3.1 can be proved. In fact, any total order on $\{1, 2, \ldots, 1', 2' \ldots\}$ that preserves $1 < 2 < \ldots$ and $1' < 2' < \ldots$ (i.e. a total order in which $\{1, 2, \ldots\}$ and $\{1', 2', \ldots\}$ are chains) will give a result. However, if we change the total order, the functions involved will no longer be Q–functions, for, unlike the supersymmetric Schur functions, the Q–functions are not independent of the total order chosen. See Example 5.1 below. Hence out next result will be defined not in terms of Q–functions but rather in terms of general generating functions for paths. In this section we assume all total orders have $\{1, 2, \ldots\}$ and $\{1', 2', \ldots\}$ as chains.

Fix a total order, ϕ, on $\{1, 2, \ldots, 1', 2' \ldots\}$. Define a shifted tableau with respect to ϕ as a shifted diagram whose boxes are filled with elements from $\{1, 2, \ldots, 1', 2' \ldots\}$ such that the entries weakly increase (with respect to ϕ) across the rows and weakly increase (with respect to ϕ) down the columns and such that the following two rules are obeyed:



1'	1	1	3	4'	5
1'	2'	2	4'	4	
1	2'	3			
2	3'				

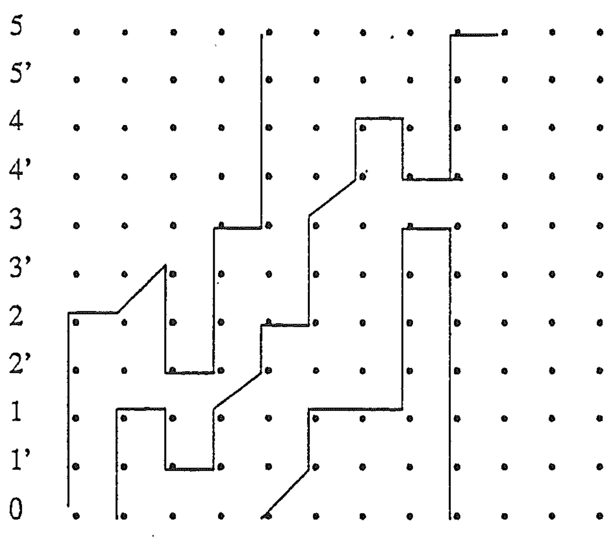


Figure 6: Example for supersymmetric Schur functions.

1) For each $k = 1, 2, 3, \ldots$, there is at most one k per column.

2) For each $k = 1, 2, 3, \ldots$, there is at most one k' per row.

Define the generating function, $\mathcal{G}^{\phi}_{\lambda/\mu}$, in the variables $X = (x_1, x_2, \ldots)$ as

$$\mathcal{G}^{\phi}_{\lambda/\mu} = \sum_{T} \prod_{\alpha \in \lambda/\mu} x_{|T(\alpha)|}, \tag{5}$$

where the summation is over all skew shifted tableaux T of shape λ/μ and with respect to ϕ, where $\alpha \in \lambda/\mu$ means α ranges over all squares in the skew shifted diagram of λ/μ, and where $|T(\alpha)| = k$ if $T(\alpha) = k$ or k'.

Example 5.1 *Example of a total order for which the corresponding tableaux do not generate Schur Q–functions. Let $\lambda = (2,1)$ with variables $x = \{x_1, x_2, x_3\}$ and with total order $\phi = 1 < 2 < 3 < 1' < 2' < 3'$. Then*

$\mathcal{G}^{\phi}_{(2,1)} = 4x_1^2x_2 + 4x_1^2x_3 + 4x_2^2x_1 + 4x_2^2x_3 + 4x_3^2x_1 + 4x_3^2x_2 + 2x_1^3 + 2x_2^3 + 2x_3^3 + 6x_1x_2x_3,$

while with total order $\psi = 1' < 1 < 2' < 2 < 3' < 3$,

$$\mathcal{G}^{\psi}_{(2,1)} = Q_{(2,1)} = 4x_1^2x_2 + 4x_1^2x_3 + 4x_2^2x_1 + 4x_2^2x_3 + 4x_3^2x_1 + 4x_3^2x_2 + 8x_1x_2x_3.\diamond$$

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ be an outside decomposition of a shifted diagram. Recall ρ is the strip consisting of the single box in position $(1,1)$. Let I be a set of coordinates of the form $(0,a)$ where $a \in \{0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, \frac{1}{2}', 1', 1\frac{1}{2}', \ldots\}$ and where $(0,a) < (0,b)$ if $a < b$ in ϕ. Define below in equation (6) a function related to the $\mathcal{G}^{\phi}$ defined in (5) but defined for an ordered pair of strips rather than for a partition.

$$\mathcal{G}^{\phi}_{(\theta_i,\theta_j)}(u_i,u_j) = \sum_{v_1 < v_2 \in I} \mathcal{G}^{\phi}_{\theta_i \# \rho}(u_i,v_1)\mathcal{G}^{\phi}_{\theta_j \# \rho}(u_j,v_2) - \mathcal{G}^{\phi}_{\theta_i \# \rho}(u_i,v_2)\mathcal{G}^{\phi}_{\theta_j \# \rho}(u_j,v_1). \tag{6}$$

where the sum extends to all pairs for which v_1 precedes v_2 in the ordering of I.

If $i = j$, note this sum is zero. Also note $\mathcal{G}^{\phi}_{(\theta_j,\theta_i)}(u_j,u_i) = -\mathcal{G}^{\phi}_{(\theta_i,\theta_j)}(u_i,u_j)$ if $i \leq j$.

Theorem 5.2 *Let λ and μ be partitions with distinct parts. Let $\theta = \theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_m$ be an outside decomposition of the shifted diagram λ/μ where θ_i, $1 \leq i \leq k$, includes a box of the main diagonal of the diagram and θ_i, $k+1 \leq i \leq m$, does not. If $2m - k$ is odd, define $\theta_0 = \emptyset$ and replace θ by $\theta_0 \cup \theta$. Then*

$$\mathcal{G}^{\phi}_{\lambda/\mu} = pf\begin{bmatrix} G & H \\ -H^T & 0 \end{bmatrix}$$

where $G = [\mathcal{G}^{\phi}_{(\theta_i,\theta_j)}(u_i,u_j)]$, $1 \leq i, j, \leq m$ for $u_i = (d-c+1, 0)$ if strip i has ending box on the top perimeter in box (c,d) of the shifted diagram or $u_i = (d-c+1, \infty)$ if strip i has ending box on the right perimeter in box (c,d) of the diagram ($u_i = (d-c+1,0)$ if both), $i = 1, \ldots, m$, and $H = [\mathcal{G}^{\phi}_{\theta_i \# \theta_j}]$, $1 \leq i \leq m$, $k+1 \leq j \leq m$.

Proof: Describe a different lattice path set-up. Label the y-axis with $\{1, 2, \ldots, 1', 2', \ldots\}$ ordered according to ϕ. Define the same kinds of steps as in Theorem 3.1 and impose the same restrictions. Now proceed as in Theorem 3.1 using a Gessel–Viennot argument to eliminate intersecting paths. $\diamond$

An example appears in Figure 7 where the total order is $1 < 2 < 1' < 3 < 4 < 2' < 3' < 5$.

Determinantal results similar to Theorem 3.4 and Corollary 3.5 can also be proved. Theorem 5.3 and Corollary 5.4 below generalize results of Okada [7] who defines his results in terms of the arbitrary total order and general generating functions. However, his results are stated only for standard shape and only for an outside decomposition into rows.

Theorem 5.3 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with distinct nonzero parts, and let $a = (a_1, a_2, \ldots, a_{l-m})$ be a strictly increasing sequence of integers chosen from $\{1, 2, \ldots, 1', 2' \ldots\}$ ordered by ϕ. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be an outside decomposition of λ/μ where $\theta_1 \ldots \theta_{l-m}$ contain some a_i, and the remaining θ_i do not. Then*

$$\mathcal{G}^{\phi}_{\lambda/\mu}(X) = \det\left[\left(x_{|a_j|}\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X^{a_j})\right)_{1\le i\le k; 1\le j\le l-m} \vdots \left(\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X)\right)_{1\le i\le k; l-m+1\le j\le k}\right],$$

with

$$\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X^{a_j}) = \begin{cases} \mathcal{G}^{\phi}_{\theta_i\#\theta_j}(x_{a_j}, x_{a_{j_1}}, \ldots) & \text{if } a_j \text{ unprimed,} \\ \mathcal{G}^{\phi}_{\theta_i\#\theta_j}(x_{a_{j_1}}, x_{a_{j_2}}, \ldots) & \text{if } a_j \text{ primed,} \end{cases}$$

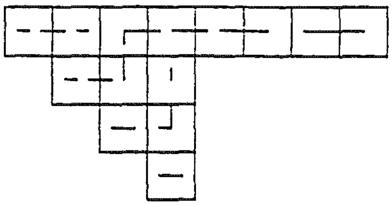
where a_{j_1} covers a_j (i.e. a_{j_1} is the element in the total order immediately after a_j) and a_{j_2} covers a_{j_1}.

Corollary 5.4 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with distinct nonzero parts, and let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be an outside decomposition of λ/μ. Then,*

$$\mathcal{G}^{\phi}_{\lambda/\mu}(X) = \sum_{a_1<a_2<\ldots<a_{l-m}} \det\left[\left(x_{|a_j|}\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X^{a_j})\right)_{1\le i\le k; 1\le j\le l-m} \vdots \left(\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X)\right)_{1\le i\le k; l-m+1\le j\le k}\right],$$

where $\mathcal{G}^{\phi}_{\theta_i\#\theta_j}(X^{a_j})$ is as in Theorem 5.3 and where the sum is over all strictly increasing sequences $a = (a_1, \ldots a_{l-m})$ of integers chosen from $\{1, 2, \ldots, 1', 2' \ldots\}$ ordered by ϕ.

Results similar to Theorem 5.3 and Corollary 5.4 but with tableaux constrained to have unprimed profiles (i.e. the arrangement for Schur P–functions) can also be proved.



1	1	2	1'	4	2'	3'	5
	2	1'	3				
		1'	2'				
			2'				

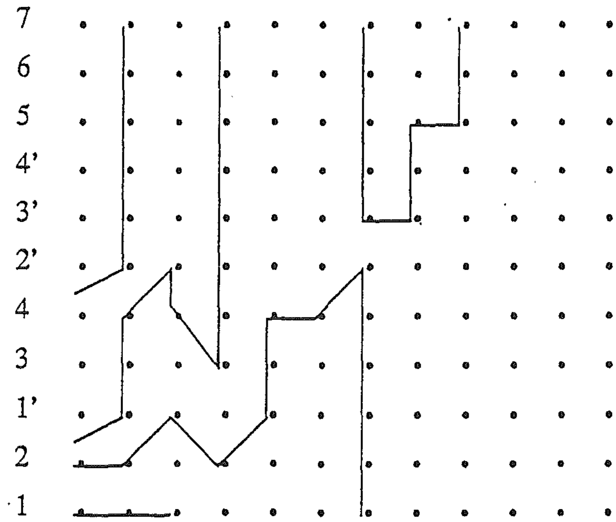


Figure 7: Example for a different total order.

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